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General scaling law in the saddle–node bifurcation: a complex phase space study

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Abstract

Saddle–node bifurcations have been described in a multitude of nonlinear dynamical systems modeling physical, chemical, as well as biological systems. Typically, this type of bifurcation involves the transition of a given set of fixed points from the real to the complex phase space. After the bifurcation, a saddle remnant can continue influencing the flows and generically, for non-degenerate saddle–node bifurcations, the time the flows spend in the bottleneck region of the ghost follows the inverse square root scaling law. Here we analytically derive this scaling law for a general one-dimensional, analytical, autonomous dynamical system undergoing a not necessarily non-degenerate saddle–node bifurcation, in terms of the degree of degeneracy by using complex variable techniques. We then compare the analytic calculations with a one-dimensional equation modeling the dynamics of an autocatalytic replicator. The numerical results are in agreement with the analytical solution.

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1. Introduction

The connection between mathematics and other scientific disciplines has become extremely relevant in recent decades since nonlinear dynamical systems theory has been applied to theoretically describe and study the dynamics governing several physical, chemical or biological systems. From the foundational studies of Henri Poincaré on the qualitative or geometrical theory of nonlinear systems of differential equations at the end of the nineteenth century, the development of a huge quantity of scientific literature based on these mathematical approaches largely justifies their use as theoretical powerful tools to tackle really complex phenomena arising in both natural and artificial systems. For instance, several works based

on nonlinear dynamical systems theory have revealed interesting results in the context of both ecological [1–4] and molecular dynamics [5–7].

Several dynamical phenomena can occur near critical points in some nonlinear dynamical systems [8–10]. For instance, delaying processes arising in the transition between two qualitatively different dynamics as the bifurcation parameter crosses the bifurcation point. Some examples of delayed transitions can be found in the vicinity of first- or second-order phase transitions, for example, the so-called *critical slowing down* [11–14], typically arising near pitchfork bifurcations [9]. Delaying phenomena can also arise near a critical point by means of the so-called *stability loss delay* [15], reported in laser physics, biophysics and chemical kinetics (see [16] for detailed references). Other dynamical properties such as *intermittency* phenomena [17–19] have been reported in the vicinity of bifurcation points. For this case, chaos is achieved via intermittency in the well-known *Mannville–Pomeau* route to chaos [8].

The dynamical behavior studied in this work appears near a saddle–node bifurcation, and is given by the so-called bottleneck or *ghost* [9]. Saddle–node ghosts involve a time delay required to pass through the bottleneck region where a saddle remnant continues influencing the flows although the two fixed points involved in the bifurcation have coalesced [9]. For non-degenerate bifurcations, this time delay, τ , typically follows the inverse square root scaling law given by

$$\tau \propto 1/\sqrt{\mu - \mu_c},$$

being μ the bifurcation parameter and μ_c the parameter value at which the bifurcation takes place. However, one of the points we address in this paper is that degenerate bifurcations give rise to different scaling laws.

Bottlenecking phenomena and its associated inverse square root scaling law have been elegantly shown in an experiment with an electronic circuit modeling Duffing’s equation [20]. This scaling law has also been described in intermittency phenomena [17–19], in a model of charge density waves [21] as well as in the theoretical framework of earlier prebiotic evolution for hypercycles [22–24]. For this system, the saddle–node bifurcation, which involves the jump of the coexistence fixed point of the hypercycle and the saddle to the complex phase space, separates the survival of these catalytic replicators from an asymptotic extinction [22–26].

Many of the mentioned models are given by polynomial or analytic differential equations. Since it is theoretically possible to look at the complex phase space (i.e., considering $x \in \mathbb{C}$), we may think of taking advantage of the information of the features of the dynamics in \mathbb{C} . In particular for the (non-degenerate) saddle–node bifurcation, two equilibrium point coalesce and apparently disappear but actually they leave the real phase space and move to the complex phase space. Consider the model for the saddle–node bifurcation

$$x' = -\epsilon - x^2.$$

For $\epsilon \neq 0$ it has two equilibrium points, $x = \pm\sqrt{-\epsilon}$, which are real for $\epsilon < 0$ but are complex for $\epsilon > 0$. It turns out that when $\epsilon > 0$ is small the complex equilibrium points are close to the origin and hence at this point the vector field (and hence the velocity of x) is small, which means that x has to spend much time to pass through the bifurcation point. This is the main reason for the so-called bottleneck or ghost phenomena. The scaling law measures the time of passage through $x = 0$ as a function of the difference of the parameter ϵ to its bifurcation value $\epsilon_c = 0$.

In this work we derive, in section 2, the scaling law for general saddle–node, one-dimensional bifurcations for an analytic differential equation using complex variable techniques and explicitly compute the leading coefficient of the corresponding asymptotic

expression. By general saddle–node bifurcation, we mean a bifurcation for which a saddle and a node coalesce in a possible degenerate way as it happens in the model $x' = -\epsilon - x^{2n}$, being the case of our study the general model given by

$$x' = a\epsilon + bx^{2n} + \text{higher order terms} \tag{1}$$

(see the precise conditions in section 2). Then, in section 3 we apply the calculations developed in section 2 to characterize the time spent in the ghost for an autocatalytic replicator. Such a passage time is then compared with numerical analysis.

2. Scaling law in the saddle–node bifurcation

We consider a one-dimensional differential equation

$$x' = f(x, \epsilon),$$

having a saddle–node bifurcation. To simplify the notation, we first assume that it takes place at $x = 0$ when $\epsilon = 0$. We also assume that f is analytic with respect to x in the neighborhood of $x = 0$. Sufficient conditions for such a bifurcation are

$$f(0, 0) = 0, \quad D_\epsilon f(0, 0) \neq 0, \tag{2}$$

and

$$D_x^j f(0, 0) = 0, \quad 1 \leq j \leq 2n - 1, \quad D_x^{2n} f(0, 0) \neq 0. \tag{3}$$

Let $a = D_\epsilon f(0, 0)$ and $b = D_x^{2n} f(0, 0)/(2n)!$. For the sake of concreteness we assume that a and b are negative, the other cases being easily obtained by changing $\epsilon \mapsto -\epsilon$ and/or $x \mapsto -x$ (as previously mentioned, the model is equation $x' = a\epsilon + bx^{2n}$). The equilibrium points are the solutions of $f(x, \epsilon) = 0$. Condition (2) allows us to apply the implicit function theorem and obtain that locally the zeros of f close to $(0, 0)$ are the points of the graph of an analytic function $\epsilon = g(x)$ such that $g(0) = 0$. Differentiating implicitly $2n$ times $f(x, g(x)) = 0$ we obtain inductively that $D^j g(0) = 0, 1 \leq j \leq 2n - 1$, and

$$D^{2n} g(0) = -\frac{D_x^{2n} f(0, 0)}{D_\epsilon f(0, 0)}.$$

This means that

$$\epsilon = g(x) = \frac{1}{(2n)!} D^{2n} g(0) x^{2n} + \dots = -\frac{b}{a} x^{2n} + O(x^{2n-1}).$$

In order to have the zeros in terms of ϵ , we just have to invert the previous expression. Except for $\epsilon = 0$, we get $2n$ complex solutions

$$\begin{aligned} x_j(\epsilon) &= \left(\frac{a\epsilon}{b}\right)^{1/(2n)} e^{i\pi(1+2j)/(2n)} + O(\epsilon^{1/n}), & \epsilon > 0, \\ x_j(\epsilon) &= \left(-\frac{a\epsilon}{b}\right)^{1/(2n)} e^{i\pi j/n} + O(\epsilon^{1/n}), & \epsilon < 0, \end{aligned} \tag{4}$$

with $0 \leq j \leq 2n - 1$, where $i = \sqrt{-1}$. Since $a/b > 0$, for $\epsilon < 0$ we have two real equilibrium points, $x_\pm(\epsilon) = \pm(-a\epsilon/b)^{1/(2n)} + O(\epsilon^{1/n})$, which correspond to $j = 0, n$, and no real equilibrium points for $\epsilon > 0$.

To decide the stability of $x_\pm(\epsilon)$ when $\epsilon < 0$, we evaluate the x -derivative of f at them. We write

$$f(x, \epsilon) = a\epsilon + \sum_{2 \leq k+l \leq 2n} c_{k,l} x^k \epsilon^l + \dots$$

Condition (3) implies $c_{k,0} = 0$ for $1 \leq k \leq 2n - 1$ and $c_{2n,0} = b$. Then

$$D_x f(x_{\pm}(\epsilon), \epsilon) = \pm 2nb(-a\epsilon/b)^{(2n-1)/(2n)} + O(\epsilon) \tag{5}$$

and therefore if $\epsilon < 0$ is small, x_+ is stable and x_- is unstable (recall that $b < 0$).

Our goal is to estimate the time needed to go from $x = \delta$ to $x = -\delta$ for some $\delta > 0$ in terms of $\epsilon > 0$ small. Since the equation is autonomous, the time is given by the integral

$$\tau_{\epsilon} = \int_{\delta}^{-\delta} \frac{dx}{f(x, \epsilon)}. \tag{6}$$

We choose $\delta > 0$ and $\epsilon_0 > 0$ small enough such that $f(x, \epsilon) \neq 0$ for $(x, \epsilon) \in [-\delta, \delta] \times (0, \epsilon_0]$. Since f is analytic we can extend it analytically to an open set U of \mathbb{C} containing $\{z \in \mathbb{C} \mid -\delta_1 \leq \operatorname{Re} z \leq \delta_1, 0 \leq \operatorname{Im} z \leq \nu\}$ for some δ_1 and ν independent of ϵ . From (4) we have that for $\epsilon > 0$ small, $f(x, \epsilon)$ has $2n$ complex zeros close to

$$x_j^*(\epsilon) = \left(\frac{a\epsilon}{b}\right)^{1/(2n)} e^{i\pi(1+2j)/(2n)}, \quad 0 \leq j \leq 2n - 1.$$

Among these x_0 and x_{n-1} are the continuation of the real equilibrium x_{\pm} points after the bifurcation which have become complex. They are simple zeros since a completely analogous computation as in (5) gives $|D_x f(x_j(\epsilon), \epsilon)| = 2n|b|(a\epsilon/b)^{(2n-1)/(2n)} + O(\epsilon) \neq 0$, if ϵ is small and are the unique zeros of f in U . Hence these zeros are simple poles of $1/f$.

The strategy to compute the integral in (6) consists of considering x as a complex variable and to apply the Residue theorem with a closed complex path of integration γ , independent of ϵ , whose intersection with the real axis coincides with the interval $[-\delta_1, \delta_1]$ (see figure 1b). By the Residue theorem, we have

$$I_{\epsilon} := \int_{\gamma} \frac{dx}{f(x, \epsilon)} = 2\pi i \sum_{0 \leq j \leq n-1} \operatorname{Res} \left(\frac{1}{f}, x_j(\epsilon) \right)$$

because the x_j with $0 \leq j \leq n - 1$ are the poles in the upper-half plane inside the path γ since they are $\epsilon^{(1)/(2n)}$ -close to $x_j^*(\epsilon)$, respectively [22].

The calculation of the residues gives

$$\begin{aligned} \operatorname{Res} \left(\frac{1}{f}, x_j(\epsilon) \right) &= \lim_{x \rightarrow x_j(\epsilon)} (x - x_j(\epsilon)) \frac{1}{f(x, \epsilon)} \\ &= \frac{1}{D_x f(x_j(\epsilon), \epsilon)} = \frac{1}{2nbx_j^{2n-1} + O(\epsilon)}. \end{aligned}$$

The sum of the residues can be estimated since its dominant part is a geometric progression

$$\begin{aligned} &\frac{1}{2nb} \left(\frac{b}{a\epsilon}\right)^{\frac{2n-1}{2n}} \sum_{j=0}^{n-1} e^{-i\pi(1+2j)\frac{2n-1}{2n}} + O(\epsilon^{-\frac{2n-2}{2n}}) \\ &= \frac{-1}{2na} \left(\frac{a}{b}\right)^{\frac{1}{2n}} \left(\frac{1}{\epsilon}\right)^{\frac{2n-1}{2n}} \sum_{j=0}^{n-1} e^{i\pi\frac{1+2j}{2n}} + O(\epsilon^{-\frac{2n-2}{2n}}) \end{aligned}$$

and the previous sum is calculated as

$$e^{i\pi/(2n)} \sum_{j=0}^{n-1} e^{i\pi j/n} = e^{i\pi/(2n)} \frac{e^{i\pi} - 1}{e^{i\pi/n} - 1} = \frac{-2}{2i \sin(\pi/(2n))}.$$

Hence we obtain

$$I_{\epsilon} = \frac{A}{\epsilon^p} + O\left(\frac{1}{\epsilon^{p-1/(2n)}}\right),$$

where

$$p = \frac{2n - 1}{2n} \quad \text{and} \quad A = \frac{1}{na} \left(\frac{a}{b}\right)^{\frac{1}{2n}} \frac{\pi}{\sin \frac{\pi}{2n}}.$$

Next we relate I_ϵ with τ_ϵ . We decompose γ as the union of the paths $\gamma_1 = [-\delta_1, \delta_1]$, γ_2 , γ_3 and γ_4 , as shown in figure 1(b). Let

$$I_\epsilon^j := \int_{\gamma_j} \frac{dx}{f(x, \epsilon)}.$$

Obviously, $I_\epsilon = \sum_{j=1}^4 I_\epsilon^j$. We will see that $I_\epsilon^2, I_\epsilon^3$ and I_ϵ^4 are bounded with respect to ϵ and therefore the $O(\epsilon^{-(2n-1)/(2n)})$ contribution to I_ϵ comes from I_ϵ^1 . We parameterize $\gamma_2(t) = \delta_1 + ivt$ with $0 \leq t \leq 1$, hence

$$I_\epsilon^2 = \int_0^1 \frac{iv dt}{f(\gamma_2(t), \epsilon)}.$$

Since $f(\gamma_2(t), \epsilon) = f(\gamma_2(t), 0) + O(\epsilon)$ and $f(\gamma_2(t), 0) \neq 0$ for $t \in [0, 1]$, we deduce

$$\frac{1}{f(\gamma_2(t), \epsilon)} = \frac{1}{f(\gamma_2(t), 0)} + O(\epsilon)$$

uniformly (by the compactness of $[0, 1]$), and therefore

$$I_\epsilon^2 = C_2 + O(\epsilon). \tag{7}$$

Analogously, we parameterize $\gamma_4(t) = -\delta_1 + iv(1 - t)$, with $0 \leq t \leq 1$, and in the same way we obtain

$$I_\epsilon^4 = C_4 + O(\epsilon). \tag{8}$$

We parameterize $\gamma_3(t) = -t + iv$ with $-\delta_1 \leq t \leq \delta_1$. Again $f(\gamma_3(t), \epsilon) = f(\gamma_3(t), 0) + O(\epsilon)$, with $f(\gamma_3(t), 0) \neq 0$ for $t \in [-\delta_1, \delta_1]$, hence $1/f(\gamma_3(t), 0) + O(\epsilon)$ uniformly and

$$I_\epsilon^3 = C_3 + O(\epsilon). \tag{9}$$

Putting everything together and taking into account (7)–(9), we have

$$\int_{\delta_1}^{-\delta_1} \frac{dx}{f(x, \epsilon)} = \frac{A}{\epsilon^p} + O\left(\frac{1}{\epsilon^{p-1/(2n)}}\right). \tag{10}$$

Given some $\delta > \delta_1$, such that $|f(x, \epsilon)| > \alpha > 0$ on $[-\delta, -\delta_1] \cup [\delta_1, \delta]$, then both

$$\int_{\delta}^{\delta_1} \frac{dx}{f(x, \epsilon)} \quad \text{and} \quad \int_{-\delta_1}^{-\delta} \frac{dx}{f(x, \epsilon)}$$

are bounded by $(\delta - \delta_1)/\alpha$ (independent of ϵ) and, therefore, the asymptotic expression of the time of passage τ_ϵ is given by the same expression (10). We remark that the dominant term in (10) is independent of δ .

If the bifurcation occurs at $x = x_c$ for the value of the parameter $\epsilon = \epsilon_c$, we only have to make the translations $x \mapsto x - x_c$ and $\epsilon \mapsto \epsilon - \epsilon_c$. Conditions (2) and (3) for the bifurcation now read $f(x_c, \epsilon_c) = 0$, $a = D_\epsilon f(x_c, \epsilon_c) \neq 0$ and $D_x^j f(x_c, \epsilon_c) = 0$, $1 \leq j \leq 2n - 1$, $b = D_x^{2n} f(x_c, \epsilon_c)/(2n)! \neq 0$, and the time of passage is (assuming $a, b < 0$)

$$\tau_\epsilon = \int_{x_c+\delta}^{x_c-\delta} \frac{dx}{f(x, \epsilon)} = \frac{A}{(\epsilon - \epsilon_c)^p} + O\left(\frac{1}{(\epsilon - \epsilon_c)^{p-1/(2n)}}\right),$$

for $\epsilon > \epsilon_c$.

In the particular non-degenerate case, $n = 1$, we have

$$\tau_\epsilon = \frac{\pi}{\sqrt{ab(\epsilon - \epsilon_c)}} + C + O(\sqrt{\epsilon - \epsilon_c}), \quad \epsilon > \epsilon_c, \quad (11)$$

where C is a constant dependent on δ but independent of ϵ .

3. Autocatalytic replicator model

The dynamics of an autocatalytic replicator can be modeled with a one-dimensional ordinary differential equation, where the state variable, $x(t)$, denotes the relative concentration of a replicator which catalyzes its own chemical formation. Actually, an autocatalytic process might also describe the qualitative dynamics of a single population of organisms with intraspecific cooperation. If we assume a well-mixed population of replicators with nonlinear, i.e., catalytic, growth and a logistic restriction function in reproduction, the population dynamics for this system can be modeled according to

$$\frac{dx}{dt} = kx^2 \left(1 - \frac{x}{c_0}\right) - \epsilon x, \quad (12)$$

where $x \in \mathbb{R}^+ \cup \{0\}$ and c_0 denotes the carrying capacity of the system. Here k and ϵ are, respectively, the intrinsic growth or self-replication rate and the density-independent decay or hydrolysis rate of the replicator. Expressing equation (12) in the form of equation (1) we obtain, by considering $\xi = x - x_c$ and $\eta = \epsilon - \epsilon_c$,

$$\frac{d\xi}{dt} = -\frac{c_0}{2}\eta - \frac{k}{2}\xi^2 - \left(\eta\xi + \frac{k}{c_0}\xi^3\right).$$

The system described by equation (12) undergoes a saddle–node bifurcation at $x_c = c_0/2$ for $\epsilon_c = c_0k/4$. The qualitative behavior as well as the transition towards extinction for an autocatalytic replicator species has been previously characterized [24]. Moreover, several studies with the well-known hypercycle model have shown that the transition between coexistence and extinction in catalytically coupled replicator species is governed by the saddle–node bifurcation [23–26]. The asymptotic extinction for these replicator systems begins once the critical decay rate ϵ_c (i.e., bifurcation point) is achieved and a saddle–node bifurcation takes place (see figure 1(a) for details). From a purely mathematical point of view, this bifurcation involves the transition from positive, real fixed points (i.e., the saddle and coexistence invariant) to complex fixed points.

The previous calculations can be applied to equation (12). For this particular case, we have $D_\epsilon f(x_c, \epsilon_c) = -c_0/2$ and $D_x^2 f(x_c, \epsilon_c) = -k$. Hence, using equation (11), the time spent to pass through the bottleneck region found in the complex phase space is given by

$$\tau_\epsilon = \frac{\pi}{\sqrt{c_0k(\epsilon - \epsilon_c)}} + C + O(\sqrt{\epsilon - \epsilon_c}), \quad \epsilon > \epsilon_c, \quad (13)$$

where C may depend on δ , k , c_0 but is independent of ϵ .

We now compare the time delay given by equation (13) with the extinction time for the autocatalytic replicator after the bifurcation, solving equation (12) numerically with the fourth-order Runge–Kutta method (using a constant time step $\delta t = 0.1$). Figure 2 shows a very good match between both time delays, especially when the distance to the bifurcation point is small, indicating that equation (11) provides a good estimate for the transient dynamics in one-dimensional ghost-induced delayed transitions occurring in the complex phase space.

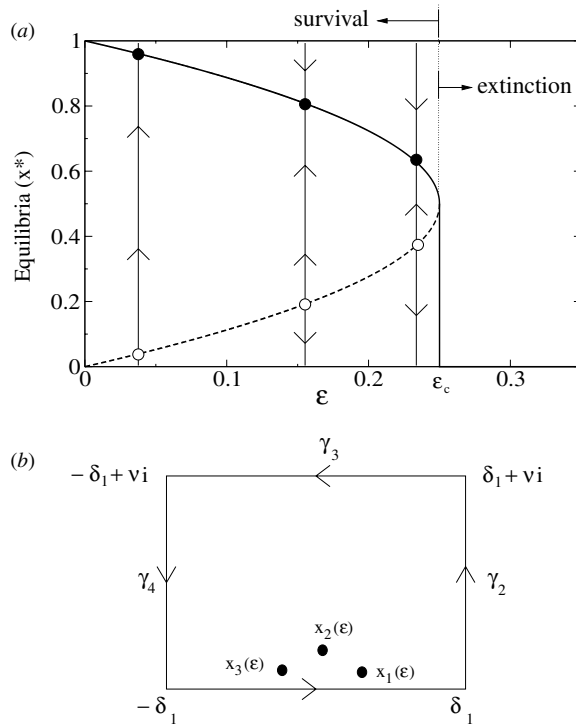


Figure 1. (a) Bifurcation diagram for equation (12) using ϵ as control parameter with $k = 1$ and $c_0 = 1$. Vertical lines are one-dimensional phase space (stable and unstable equilibria are, respectively, indicated with black and white circles, both placed on the solid and on the dashed branches of the diagram). Arrows indicate the direction of the flow on the line. (b) Schematic diagram of the complex path of integration γ for $n = 3$.

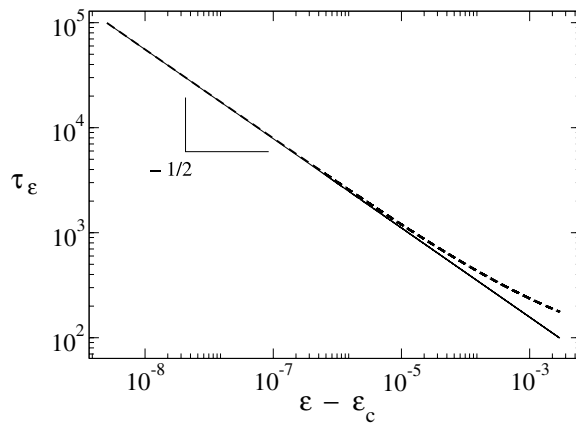


Figure 2. Time delay, τ_ϵ , near bifurcation threshold in a doubly logarithmic scale. Dashed thick line displays the delay predicted by equation (13), and the solid line shows the delay numerically obtained from equation (12) with $k = 1$, $c_0 = 1$ and $x(0) = 0.5$. We assume replicator's extinction with $[x] < 10^{-8}$.

4. Conclusion

We have analytically derived the scaling law governing the trajectories time passage through the ghost for a general one-dimensional, analytical, autonomous dynamical system undergoing a not necessarily non-degenerate saddle–node bifurcation where a stable and an unstable fixed point collide in phase space and jump into the complex phase space, in terms of the degree of degeneracy. The time delay near the bifurcation threshold obtained analytically perfectly fits with numerical solutions for the autocatalytic replicator equation.

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